dominated by noise. Hence a mask is defined which can be put on the image and thus noise can be removed without losing the fine details of the true image.

The use of wavelet packages is also discussed. Usually, the wavelet transform decomposes a signal in a high- and a low-frequency component. The low-frequency part is again decomposed in two parts and so on. However, when splitting the high-frequency part as well, one obtains a redundant transform from which an optimal basis can be chosen to represent the image. All these techniques are illustrated in practical applications of X-ray computer tomography, magnetic resonance imaging, positron emission tomography, mammography, and many more.

Part III deals with biomedical signal processing. These are typically one-dimensional signals, in general time-varying, nonstationary, sometimes transient, and, again, corrupted by noise. We give a sample of the wavelet transform applications in this domain.

The excellent time localization property of wavelets is used to find several phenomena in a signal which occur at different frequencies and localize these events in time. For certain stochastic processes, such as action potentials or human heartbeat times, it is essential to estimate the fractal exponent of the process. Here again the wavelets are shown to outperform the Fourier transform. Furthermore, the continuous complex wavelet transform is used to analyze electrocardiograms. The modulus maxima and the $\pm\pi/2$ phase crossing show the position of sharp signal transitions while modulus minima correspond to flat segments of the signal. In microvascular pulmonary pressure observations, two signals interfere. Here the signals are separated by using filtering techniques based on wavelets.

Part IV uses wavelets for mathematical models in biology. The multiresolution structure of the continuous wavelet transform corresponds to a natural human perception of sounds. Therefore wavelets are well suited to make auditory nerve models. To measure blood velocity, traditional methods are based on the Doppler effect when the movement of reflecting particles in the bloodstream are measured. It is illustrated here how the wideband wavelet transform gives a viable alternative. Event-related potentials are reactions of the brain to certain stimuli. Analysis of such signals is typically done by principal component analysis. However, it is shown that wavelets, due to their locality, allow to the analysis of such signals effectively. When using *a priori* information, the data can be drastically reduced. Also the structure of macromolecules can be deduced from a wavelet analysis of the energy function. Here the multiresolution of wavelets allows for the grouping of certain molecules. This technique can also be used to represent complex surfaces, like for example in computer tomography. This in a sense closes the circle in this wide variety of applications that are presented in this volume.

The book is of great importance for researchers working in medical or biological signal and image analysis. They will learn about wavelet alternatives for classical approaches. The wavelet researcher will certainly gain by learning about the particular problems posed by the applications of this particular, yet important field of wavelet based analysis.

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Th. M. Rassias and J. Šimša, *Finite Sums Decompositions in Mathematical Analysis*, Wiley, Chichester, 1995, vi + 172 pp.

This is a wonderful, well-written little book which was inspired by the simple question: which (scalar-valued) functions $h = h(x, y)$ of two variables $x$ and $y$ have a representation in the form

$$h(x, y) = \sum_{i=1}^{n} f_i(x)\, g_i(y) \qquad (1)$$

for some (scalar-valued) functions $f_i$ and $g_i$? (By scalar-valued, we mean real or complex-valued.) More generally, they consider the question, posed by Th. Rassias (1989), of which functions $h$ of $k \geqslant 2$ variables can be represented as the sum of products of functions $f_{i1}, ..., f_{ik}$ of a single variable:

$$h(x_1, x_2, ..., x_k) = \sum_{i=1}^{n} f_{i1}(x_1) f_{i2}(x_2) \cdots f_{ik}(x_k). \tag{2}$$

The question of representing functions of several variables by means of functions of fewer variables has captivated the imagination of mathematicians for centuries. For example, one such result is due to A. Kolmogorov (1957) and may be stated as follows. Each continuous function $h$ on the $k$-dimensional cube can be represented in the form

$$h(x_1, x_2, ..., x_k) = \sum_{i=1}^{2k+1} \varphi_i \left[ \sum_{j=1}^{k} \alpha_{i,j}(x_j) \right]$$

for some continuous functions $\varphi_i$ and $\alpha_{i,j}$. Moreover, the functions $\alpha_{i,j}$ can be chosen *independently* of $h$.

Another example, which is closer to the subject matter of this book, is due to J. d'Alembert (1747), who essentially proved that if a sufficiently smooth function $h$ has the representation

$$h(x, y) = f(x)\, g(y), \tag{3}$$

then

$$\begin{vmatrix} h & h_y \\ h_x & h_{xy} \end{vmatrix} = 0. \tag{4}$$

More generally, C. Stéphanos announced at the Third International Congress of Mathematicians (1904) that (sufficiently smooth) functions $h$ having the representation (1) must be a solution of the "Wronskian" partial differential equation of order $n+1$:

$$\det W_{n+1} h(x, y) := \begin{vmatrix} h & h_y & \cdots & h_{y^n} \\ h_x & h_{xy} & \cdots & h_{xy^n} \\ \vdots & \vdots & \cdots & \vdots \\ h_{x^n} & h_{x^n y} & \cdots & h_{x^n y^n} \end{vmatrix} = 0. \tag{5}$$

Th. Rassias (1986) gave an example showing that the converse of this result is false. In fact, his function $h$ satisfies Eq. (4) but not (3).

F. Neuman (1981) clarified the functional equation (5) as follows. Suppose $I$ and $J$ are two intervals in $\mathbb{R}$ and the scalar function $h$ on $I \times J$ has a continuous partial derivative $h_{x^n y^n}$. If $h$ has the form (1), then

$$\det W_{n+1} h(x, y) = 0 \qquad \text{for every} \quad (x, y) \in I \times J. \tag{6}$$

Conversely, if $h$ satisfies (6) and

$$\det W_n h(x, y) \neq 0 \qquad \text{for every} \quad (x, y) \in I \times J, \tag{7}$$

then $h$ has the representation (1) on $I \times J$.

The *Casorati determinant of order n* for the scalar function $h$ on $X \times Y$ is defined on $X \times Y$ by

$$\det C_n h(x, y) := \begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) & \cdots & h(x_1, y_n) \\ h(x_2, y_1) & h(x_2, y_2) & \cdots & h(x_2, y_n) \\ \vdots & \vdots & \cdots & \vdots \\ h(x_n, y_1) & h(x_n, y_2) & \cdots & h(x_n, y_n) \end{vmatrix},$$

where $x = (x_1, x_2, ..., x_n) \in X^n$ and $y = (y_1, y_2, ..., y_n) \in Y^n$.

F. Neuman (1981) also established the following result. If $h$ is of the form (1) on $X \times Y$, then

$$\det C_{n+1} h(x, y) = 0 \qquad \text{for every} \quad x \in X^{n+1} \text{ and } y \in Y^{n+1}. \tag{8}$$

Conversely, if $h$ satisfies (8) and if

$$\det C_n h(x^0, y^0) \neq 0 \qquad \text{for some} \quad x^0 \in X^n \text{ and } y^0 \in Y^n,$$

then $h$ has a representation (1) on $X \times Y$.

These theorems indicate the important role played by certain determinants—namely, the Wronskian and Casorati determinants—in establishing representation theorems of type (1). The book devotes the entire first chapter to developing the basic properties of these determinants.

In Chapter 2, the basic representation theorems for functions of two variables are presented, including both of Neuman's results stated above.

In Chapter 3, the problem of determining necessary and sufficient conditions in order that a function $h$ of three variables have a representation in the form (2) (with $k = 3$) is considered. It turns out that a straightforward application of the method used for two variables in Chapter 2 does not seem to work. One difficulty appears to be in trying to define the determinant of a three-dimensional matrix $A = [a_{ijk}]_{i,j,k=1}^n$.

The purpose of Chapter 4 is to develop a method for finding a system of linear partial differential equations whose solution set forms a prescribed finite-dimensional linear space of smooth functions in several variables. The results are then used in Chapter 5 for attacking the general representation (1), where now, more generally, $x$ and $y$ are allowed to be two *vector* (rather than scalar) variables.

In Chapters 6 and 7, the main question considered is this: If the function $h$ of two variables does *not* have a representation of the form (1), what is the next best thing we can strive for? Specifically, let $X \times Y$ be the Cartesian product of measure spaces $X$ and $Y$, $h \in L_2(X \times Y)$, $n \geqslant 1$ be a integer, and

$$\mathscr{S}_n := \left\{ \sum_{i=1}^n f_i g_i \mid f_i \in L_2(X), g_i \in L_2(Y) \right\}.$$

Since $h \notin \mathscr{S}_n$, we seek then the best approximation to $h$ from $\mathscr{S}_n$. We should note that $\mathscr{S}_n$ is *not* convex in general (except in the trivial case when $X$ or $Y$ is a singleton). Thus the general theory of best approximation from a closed convex subset of a Hilbert space is of no help here.

The authors establish an existence theorem, a characterization theorem, a distance formula, and a theorem which characterizes when the best approximation is unique. The results are related to Hilbert–Schmidt decompositions. Some of the results of Chapter 7 can also be found in a proceedings article of M. Golomb [Approximation by functions of fewer variables, *in On Numerical Approximation* (R. E. Langer, Ed.), pp. 275–328, Univ. Wisconsin Press, Madison, 1959]. Some of the proofs given in Chapter 7 and in Golomb (1959) are different, and each approach seems to have its own merit.

In the last chapter (which was written in collaboration with A. Prástaro), the authors return to the d'Alembert equation (4), and they make a detailed study of it from the point of view of the modern geometric theory of partial differential equations. This chapter, unlike the earlier ones which require only a background subsumed in the usual graduate course in real analysis, requires more mathematical sophistication. Terms like tensor products, $n$-dimensional manifolds, tangent bundles, cotangent bundles, the tangent space at a point, fibre bundles, etc., appear in Chapter 8 for the first time. The reader not versed in differential geometry may find the going a little rough here.

The book concludes with a section stating seven open problems.

I found this to be a well-written book with a unified approach to a subject whose main results had heretofore only been located in journal articles and unpublished manuscripts. I can heartily recommend it to anyone who is interested in knowing when a function of many variables can be represented as the sum of a product of functions of single variables.

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Ahmed I. Zayed, *Handbook of Function and Generalized Function Transformations*, CRC Press, Boca Raton, FL, 1996, xxii + 643 pp.

This volume completes a collection of books devoted to various classes of one- and two-dimensional integral transformations which are used as important mathematical tools for solving problems in all areas of physical sciences.

As is known in the theory of integral transformations, the Laplace, Fourier, and Mellin transforms play an important role, not only in solving many problems in applications, but also in the composition structure and mapping properties of other integral transforms. This means that many integral transformations were constructed as a composition of the classical transforms mentioned above and some substitution operator.

The author gives useful information about the theory and applications of these transforms based on classical books of R. Churchill, I. Sneddon, E. C. Titchmarsh, and A. Zemanian. This includes mapping properties in the classical function spaces as well as in spaces of generalized functions, existence theorems, inversion formulas, the Parseval relation, convolution properties, and applications in finding classical and generalized solutions of different classes of differential and integral equations.

The author made this book self-contained and included all preliminary material from real, complex, and functional analysis, generalized functions and Schwartz distributions, theory of special functions and orthogonal polynomials. This material is contained in Chapters 1–4. Further parts consist of a collection of linear and non-linear (the Zak transform) transformations. This includes the category of integral transformations known as *Mellin convolution type transforms*: the Hankel, Stieltjes, Hilbert, Meijer, Hartley, Mittag–Leffler, Weierstrass, Abel, $Y$-, $I$-transforms, the Riemann–Liouville and Weyl fractional integrals, the hypergeometric transforms, the $G$-, $H$-, and $E$-transforms. The essential part of the book is devoted to *the Fourier-type transforms*, the discrete transforms, the wavelet transforms, and the Radon transforms and to the important class of the integral transformations that depend upon the parameter or index of a hypergeometric function in the kernel (*index transforms*) such as the Kontorovich–Lebedev transform, the Mehler–Fock transform, and the index ${}_2F_1$-transform.

It should be noted that over the past 30 years the theory of integral transformations has been intensively developed and various methods and approaches of obtaining new transforms and their inversions have been discovered, which has resulted in increased investigation of the new properties for known integral transformations. We mention for example the composition